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LETTER TO THE EDITOR

Intelligent states for the Anandan–Aharonov parameter-based uncertainty relation

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Abstract. We obtain the form of the intelligent states for the recent parameter-based uncertainty relation of Anandan and Aharonov.

Intelligent states are quantum states which satisfy the equality in the uncertainty relation for non-commuting observables [1]. Recently, Anandan and Aharonov [2] proved the parameter-based uncertainty relation (PBUR):

$$\langle \Delta E \rangle_t \delta t \geqslant \frac{h}{4} \tag{1}$$

where δt is the distance of translation to the 'nearest' orthogonal state, $\delta t = \inf\{t > 0 | \langle \psi | \hat{U}(t) | \psi \rangle = 0\}$, $\hat{U}(t)$ is the time evolution operator, and $\langle \Delta E \rangle_t = 1/\delta t \int_0^{\delta t} \Delta E(t)\, \mathrm{d}t$ is the time average of the energy uncertainty $((\Delta E)^2 = \langle \psi | \hat{H}^2 | \psi \rangle - \langle \psi | \hat{H}^2 | \psi \rangle^2)$.

A similar PBUR for position and momentum was derived by Yu [3]. In this work we discuss the intelligent states for PBUR.

The proof of (1) is based on a geometric argument: the time evolution of the state ψ (between orthogonal states, if they exist) gives a curve $\gamma(t)$ on the unit sphere $S(H)$ where H is the relevant Hilbert space. Consider the natural projection $\pi : S(H) \to P(H)$, where $P(H)$ is the projective Hilbert space. The length, measured in $P(H)$, of the curve $\pi(\gamma)(t)$, is obviously not smaller than the distance between the initial and the final states, which, by definition is measured along a geodesic in $P(H)$. The left-hand side of (1), as shown in [2], is the length of $\gamma(t)$ (divided by $\frac{\hbar}{2}$). Equality in (1) holds only for states that evolve on a geodesic in $P(H)$. We refer to such states as intelligent states, in analogy to [1].

The inequality (1) holds also if $\hat{H}$ is replaced by any other Hermitian operator $\hat{A}$, and t is replaced by a parameter (say φ) that describes evolution of the system by the action of the unitary operator $\hat{U}_A(\varphi) = \exp(\frac{\mathrm{i}\varphi}{\hbar}\hat{A})$. (As examples, $\hat{A}$ may be the number operator $\hat{N}$, or the z-component of angular momentum $\hat{j}_z$, or the momentum operator [3].) The uncertainty relation (1) becomes

$$\Delta \hat{A} \delta \varphi \geqslant \frac{h}{4} \tag{2}$$

where $\Delta \hat{A}$ and $\delta \varphi$ are defined analogously to ΔE and δt.

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Assume that $\hat{A}$ (independent of φ) has a complete basis of normalized eigenstates $\{|\psi_\alpha\rangle\}_{\alpha\epsilon I}$ with non-degenerate eigenvalues $\{m_\alpha\}_{\alpha\epsilon I}$, with I a set of indices. We prove now that all states of the form

$$|\psi\rangle = c_1|\psi_\alpha\rangle + c_2|\psi_\beta\rangle, \alpha \neq \beta; |c_1|^2 = |c_2|^2 = \tfrac{1}{2} \tag{3}$$

are intelligent states and there are no others.

Consider the two-dimensional subspace span$\{|\psi_\alpha\rangle, |\psi_\beta\rangle\}$. To obtain the vanishing of $\langle\psi|\exp(\frac{i\varphi}{\hbar}\hat{A})|\psi\rangle$ (for $|\psi\rangle = c_1|\psi_\alpha\rangle + c_2|\psi_\beta\rangle$) we need to find a solution of the equation:

$$|c_1|^2 \exp\left(\frac{i\varphi}{\hbar}m_\alpha\right) + |c_2|^2 \exp\left(\frac{i\varphi}{\hbar}m_\beta\right) = 0. \tag{4}$$

Hence

$$\exp\left(\frac{i\varphi}{\hbar}(m_\alpha - m_\beta)\right) = -\frac{|c_1|^2}{|c_2|^2} \Rightarrow |c_1|^2 = |c_2|^2 = \frac{1}{2}. \tag{5}$$

Therefore $\exp(\frac{i\delta\varphi}{\hbar}(m_\alpha - m_\beta)) = -1$ and $\delta\varphi = \frac{\pi\hbar}{|m_\alpha - m_\beta|}$.

An elementary calculation shows that for all states of the form (3) $\Delta\hat{A}$ is maximal when $|c_1|^2 = |c_2|^2$. However, for any other states of the form (3), while $\Delta\hat{A}$ is smaller, $\delta\varphi$ is infinite! (because there does not exist a solution of equation (4) for $|c_1| \neq |c_2|$). It is now easy to see that for the states (3) the inequality becomes an equality. Indeed,

$$\Delta\hat{A}^2 = \frac{1}{2}m_\alpha^2 + \frac{1}{2}m_\beta^2 - \left(\frac{1}{2}m_\alpha^2 + m_\beta)\right)^2 = \left(\frac{m_\alpha - m_\beta}{2}\right)^2 \tag{6}$$

$$\Delta\hat{A}\delta\varphi = \frac{\pi\hbar}{|m_\alpha - m_\beta|} \cdot \frac{|m_\alpha - m_\beta|}{2} = \frac{h}{4}. \tag{7}$$

This proves that states (3) are indeed intelligent states and in the two-dimensional subspace there are no other intelligent states.

To prove that any state not of the form (3) is not an intelligent state we use the following argument: a geodesic in $P(H)$ is an image under the projection π of some geodesic in $S(H)$ (where both $P(H)$ and $S(H)$ are treated as real manifolds). It is well known that the geodesics on the unit sphere are intersections of two-dimensional (real) subspaces V with the sphere [4]. Therefore, geodesics lie in a two-dimensional subspace V.

Let $|\psi\rangle = \sum_{\alpha\epsilon I} c_\alpha|\psi_\alpha\rangle$ be a general state, where there exist at least three α's such that $c_\alpha \neq 0$, say $\alpha_1, \alpha_2, \alpha_3$. Suppose $|\psi\rangle$ is an intelligent state, then $\exp(\frac{i\varphi}{\hbar}\hat{A})|\psi\rangle\epsilon V$, $\dim(V) = 2$. Hence, there is a state $|\Phi\rangle$, which is orthogonal to V and has a non-zero component in span$\{|\psi_{\alpha_1}\rangle, |\psi_{\alpha_2}\rangle, |\psi_{\alpha_3}\rangle\}$

$$\langle\Phi|\sum_{\alpha\epsilon I} c_\alpha \exp\left(\frac{i\varphi}{\hbar}\hat{A}\right)|\psi_\alpha\rangle = 0. \tag{8}$$

Hence

$$\sum_{\alpha\epsilon I} c_\alpha\langle\Phi|\psi_\alpha\rangle \exp\left(\frac{i\varphi}{\hbar}m_\alpha\right) = 0. \tag{9}$$

By applying to both sides $\int_{-\infty}^{\infty}(\cdot)\exp(m_{\alpha_j}\frac{i\varphi}{\hbar})\,d\varphi$

$$\sum_{\alpha\epsilon I} c_\alpha\langle\Phi|\psi_\alpha\rangle \int_{-\infty}^{\infty} \exp\left(\frac{i\varphi}{\hbar}(m_{\alpha_j} - m_\alpha)\right) d\varphi = 0 \tag{10}$$

$$\sum_{\alpha\epsilon I} c_\alpha\langle\Phi|\psi_\alpha\rangle\delta(m_{\alpha_j} - m_\alpha) = 0 \tag{11}$$

$$\Rightarrow \langle\Phi|\psi_{\alpha_j}\rangle c_{\alpha_j} = 0 \Rightarrow \ni j\epsilon\{1, 2, 3\};\ c_{\alpha_j} = 0 \tag{12}$$

contradiction. The proof is completed.

To conclude, we add that if $\hat{A}$ has degeneracies, a similar proof shows that the intelligent states are states of the form $|\psi\rangle = c_1|\psi_\alpha\rangle + c_2|\psi_\beta\rangle$, where $m_\alpha \neq m_\beta$, and $|c_1|^2 = |c_2|^2 = \frac{1}{2}$.

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